***MATCH***
*Communications in Mathematical and in Computer Chemistry*

MATCH Commun. Math. Comput. Chem. **62** (2009) 561-572

ISSN 0340 - 6253

# INCIDENCE ENERGY OF A GRAPH

**Mohammadreza Jooyandeh,**[a] **Dariush Kiani**[a,b]
and **Maryam Mirzakhah**[a]

[a]*Faculty of Mathematics and Computer Science,*
*Amirkabir University of Technology,*
*P. O. Box 15875-4413, Tehran, Iran.*

[b]*Institute for Studies in Theoretical Physics*
*and Mathematics (IPM), Tehran, Iran.*
e-mail: mohammadreza@jooyandeh.info ;
dkiani@aut.ac.ir ;
maryam@mirzakhah.info

(Received October 16, 2008)

## Abstract

The energy of a graph $E(G)$, is the sum of the singular values of its adjacency matrix. We define *incidence energy* of the graph $G$, denoted by $IE(G)$, as the sum of the singular values of its incidence matrix. We are interested to find the relation between the energy and the incidence energy of graphs. For any graph $G$ we obtain a bipartite graph $\widehat{G}$ such that $IE(G) = \frac{E(\widehat{G})}{2}$. Moreover we find some similar upper and lower bounds of energy for incidence energy. Finally we show that for any proper subgraph $H$ of the graph $G$, $IE(G) > IE(H)$.

## INTRODUCTION

The energy of a graph is defined as the sum of the absolute values of its eigenvalues. This concept was proposed quite some time ago by Gutman in [1], motivated by (much older) chemical applications [2–5]. Research on graph energy is nowadays very active, as seen from the recent papers [6–19] and the references quoted therein.

Let $G$ be an undirected, simple and finite graph with $n$ vertices and $m$ edges, the vertex set and the edge set of $G$ are denoted by $V(G)$ and $\mathrm{E}(G)$, respectively. Also the adjacency matrix and the incidence matrix of the graph $G$ are denoted by $A(G)$ and $I(G)$, respectively. Suppose $E$ is a subset of $\mathrm{E}(G)$, the spanning subgraph of $G$ with edge set $\mathrm{E}(G)\setminus E$ is denoted by $G\setminus E$. Also let $H$ be a subgraph of $G$, then $G\setminus H$ denotes the spanning subgraph of $G$ whose edge set is $\mathrm{E}(G) \setminus \mathrm{E}(H)$. The star, path and complete graph with $n$ vertices will be denoted by $S_n$, $P_n$, and $K_n$, respectively. Also the complete bipartite graph with the partitions of size $r$ and $s$ is denoted by $K_{r,s}$.

Let $A$ be any $n$ by $m$ matrix with real entries. The singular values of the matrix $A$ are the square roots of the eigenvalues of $AA^t$, where $A^t$ is the transpose of $A$. Also the eigenvalues of the square matrix $B$ of order $n$, are denoted by $\lambda_1(B), \ldots, \lambda_n(B)$, where are arranged in non-increasing order. If $A$ is a symmetric matrix, then its singular values are the absolute values of its eigenvalues. So the energy of a graph $G$ is indeed the sum of the singular values of its adjacency matrix [20]. Nikiforov in [20] has extended the concept of graph energy for arbitrary matrices. More precisely for any $n \times m$ matrix $A$, the energy of $A$ is defined as the sum of its singular values.

Let $\sigma_1(G), \ldots, \sigma_n(G)$ be the singular values of the incidence matrix of a graph $G$, now we define $IE(G) := \sum\limits_{i=1}^{n} \sigma_i(G)$, which is called the *incidence energy* of $G$. It is clear that $IE(G) \geq 0$ and the equality holds if and only if $G$ has no edges. Also if the graph $G$ has components $G_1, \ldots, G_c$, then $IE(G) = \sum\limits_{i=1}^{c} IE(G_i)$.

## ELEMENTARY RESULTS

It is well-known that for a graph $G$, $I(G)I(G)^t = A(G) + diag(d_1, \dots, d_n)$, while $diag(d_1, \dots, d_n)$ is the diagonal valency matrix of $G$. Therefore $\sum\limits_{i=1}^{n} \sigma_i(G)^2 = 2m$.

**Theorem 1** *Let $G$ be a graph, then $IE(G) = \frac{E(\widehat{G})}{2}$, in which $\widehat{G}$ is the bipartite graph with adjacency matrix*

$$\left[ \begin{array}{cc} 0 & I(G) \\ I(G)^t & 0 \end{array} \right]. \tag{1}$$

**Proof.** Suppose $X$ is the matrix that is obtained by adding some zero rows or columns to $I(G)$ in order to make a square matrix.

By [21, Problem 34.8] the eigenvalues of the matrix $\left[ \begin{array}{cc} 0 & X \\ X^t & 0 \end{array} \right]$ are $\sigma_1(G), \dots, \sigma_n(G), 0, \dots, 0, -\sigma_n(G), \dots, -\sigma_1(G)$. Also the matrix (1) has the same non-zero eigenvalues. □

Therefore Theorem (1) gives a relation between the incidence energy of a graph $G$ and the energy of the graph $\widehat{G}$, where the graph $\widehat{G}$ is the bipartite graph which is obtained from $G$ by adding a vertex on each edge of $G$.

**Remark 2** *If the energy of a graph is rational, then it must be an even number (see [22]). So Theorem (1) and this property of the graph energy conclude that if the incidence energy of a graph is rational, then it must be an integer number.*

**Proposition 3** *The incidence energy of a graph cannot be an odd number.*

**Proof.** According to the Theorem (1), $\sigma_i(G)$ is an eigenvalue of the graph $\widehat{G}$. So by [23, Lemma 1, 2, 3], $\sum\limits_{i<j} \sigma_i(G)\, \sigma_j(G)$ is integer if it is a rational number. Therefore by

$$IE(G)^2 = \sum_i \sigma_i^2(G) + 2 \sum_{i<j} \sigma_i(G)\, \sigma_j(G)$$

we are done. □

**Theorem 4** *Let $G$ be a graph, then $IE(G) \geq rank(I(G))$.*

**Proof.** According to [24], $E(G) \geq rank(G)$, where $rank(G)$ is the rank of the adjacency matrix of $G$. By combining the Theorem (1) and this inequality, we have $IE(G) \geq \frac{rank(\widehat{G})}{2} = rank(I(G))$. □

Let $G$ be any connected graph. If $G$ is bipartite, $rank(I(G)) = n - 1$ otherwise $rank(I(G)) = n$ (see [25]). Therefore for any connected graph $G$, $IE(G) \geq n - 1$. Moreover if $G$ is not bipartite, $IE(G) \geq n$.

## UPPER AND LOWER BOUNDS FOR THE INCIDENCE ENERGY

It is easy to show that $\sqrt{\sum_i \sigma_i(G)^2} \leq \sum_i \sigma_i(G)$ and the equality holds if and only if at most one of the $\sigma_i(G)$ is non-zero.

**Theorem 5** *Let $G$ be a graph of order $n$ with $m$ edges, then $\sqrt{2m} \leq IE(G) \leq \sqrt{2mn}$. Moreover, the left equality holds if and only if $m \leq 1$. On the other hand the right equality holds if and only if $m = 0$.*

**Proof.** According to the above statement, the left inequality is obvious. Also for the equality case, $rank(I(G)I(G)^t)) \leq 1$. So $rank(I(G)I(G)^t)) = rank(I(G))$ which leads to $G$ must have at most one edge (if the graph $G$ has more than one edge, clearly $rank(I(G)) > 1$).

For the right side, by applying the Cauchy-Schwartz inequality, the following would be obtained

$$IE(G) = \sum_{i=1}^{n} \sigma_i(G) \leq \sqrt{n \cdot \sum_{i=1}^{n} \sigma_i(G)^2} = \sqrt{2mn},$$

and the equality is attained if and only if $(\sigma_1(G), \ldots, \sigma_n(G))$ and $(1, \ldots, 1)$ are linearly dependent. So $\sigma_i(G)^2 = l$ for all $i = 1, \ldots, n$, where $l$ is a rational number and $nl = 2m$. Thus there exists unitary matrix $P$ such that $PI(G)I(G)^t P^{-1} = lI$ and consequently $I(G)I(G)^t = lI$. So $A(G) = 0$, $diag(d_1, \ldots, d_n) = lI$, and finally we have $l = 0$. □

## INCIDENCE ENERGY OF A GRAPH AND ITS SUBGRAPHS

In this part, the incidence energy of a graph and its subgraphs one are compared. We start with some concepts of matrix theory.

Let $A$ and $B$ be complex matrices of order $r$ and $s$, respectively ($r \geq s$). The eigenvalues of $B$ interlace the eigenvalues of $A$, if $\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{r-s+i}(A)$ for $i = 1, \ldots, s$.

**Theorem 6** *[26, p.8] If* $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^t & A_{22} \end{bmatrix}$ *is a symmetric matrix, then the eigenvalues of* $A_{11}$ *interlace the eigenvalues of* $A$.

**Theorem 7** *[27, p.51] If* $A$ *and* $B$ *are real symmetric matrices of order* $n$ *and* $C = A + B$*, then*

$$\begin{aligned} \lambda_{i+j+1}(C) &\leq \lambda_{i+1}(A) + \lambda_{j-1}(B) \\ \lambda_{n-i-j}(C) &\geq \lambda_{n-i}(A) + \lambda_{n-j}(B) \end{aligned}$$

*for* $i, j = 0, \ldots, n$ *and* $i + j \leq n - 1$.

In particular, for all integer $i$ ($1 \leq i \leq n$),

$$\lambda_i(C) \geq \lambda_i(A) + \lambda_n(B) \ . \tag{2}$$

The following theorem shows that the incidence energy of a graph is greater than its proper subgraphs one.

**Theorem 8** *Let* $G$ *be a graph and* $E$ *be a non-empty subset of* $E(G)$*, then* $IE(G) > IE(G \setminus E)$.

**Proof.** Let $H$ be the spanning subgraph of $G$ such that $\mathrm{E}(H) = E$. The incidence matrix of $G$ can be partitioned as $I(G) = \begin{bmatrix} I(H) & I(G \setminus E) \end{bmatrix}$, and so $I(G)I(G)^t = I(H)I(H)^t + I(G \setminus E)I(G \setminus E)^t$. Since $I(H)I(H)^t$ is positive semi-definite, by Eq. (2), we have $\lambda_i(I(G)I(G)^t) \geq \lambda_i(I(G \setminus E)I(G \setminus E)^t)$ ($i = 1, \ldots, n$) and it follows that $IE(G) \geq IE(G \setminus E)$.

Moreover, $\lambda_i(I(G)I(G)^t) = \lambda_i(I(G \setminus E)I(G \setminus E)^t)$ for all $i$ ($i = 1, \ldots, n$), if the equality holds. Consequently, $trace(I(G)I(G)^t) = trace(I(G \setminus E)I(G \setminus E)^t)$ and it implies that $trace(I(H)I(H)^t) = 0$. Since $I(H)I(H)^t$ is positive semi-definite, $\lambda_i(I(H)I(H)^t) = 0$ $(i = 1, \ldots, n)$. It follows that $I(H) = 0$. $\square$

According to $IE(K_n) = \sqrt{2(n-1)} + (n-1)\sqrt{n-2}$, the following corollary is obvious.

**Corollary 9** *Let $G$ be a non-empty graph with clique number $c$. Then $IE(G) \geq \sqrt{2(c-1)} + (c-1)\sqrt{c-2}$. In particular, if $G$ has at least one edge then $IE(G) \geq \sqrt{2}$.*

**Corollary 10** *Among all graphs with $n$ vertices, the complete graph $K_n$ is the only graph with maximum incidence energy.*

**Proposition 11** *Let $G$ be any graph and $e \in E(G)$, then*

$$IE(G \setminus \{e\}) - \sqrt{2} \leq IE(G) \leq IE(G \setminus \{e\}) + \sqrt{2}\ .$$

**Proof.** Obviously $\widehat{G \setminus \{e\}} = \widehat{G} \setminus K_{1,2}$. In [28] it is shown that if $H$ is an induced subgraph of the graph $G$, then

$$E(G \setminus H) - E(H) \leq E(G) \leq E(G \setminus H) + E(H)\ .$$

Now let $H = K_{1,2}$, by combination of this inequality and Theorem (1) we are done. $\square$

Note that the following theorem improves the left side of the inequality of the Proposition (11).

**Theorem 12** *Let $G$ be a connected graph and $e$ be an edge of $G$. Then*

$$IE(G) \geq \sqrt{IE(G \setminus \{e\})^2 + 2}\ . \tag{3}$$

*Moreover, the equality holds if and only if $G = K_2$.*

**Proof.** If $e \in \mathrm{E}(G)$, the incidence matrix of the graph $G$ can be represented in the form of

$$I(G) = \begin{bmatrix} I(G \setminus \{e\}) & u \end{bmatrix}$$

where $u$ is a vector of size $n$ whose the first two components are 1 and the others are 0. Therefore

$$I(G)I(G)^t = I(G \setminus \{e\})I(G \setminus \{e\})^t + \begin{bmatrix} J_2 & 0 \\ 0 & 0 \end{bmatrix}$$

where $J_2$ is the all ones matrix of order 2. This implies that

$$trace(I(G)I(G)^t) = 2 + trace(I(G \setminus \{e\})I(G \setminus \{e\})^t) \ . \tag{4}$$

On the other hand Eq. (2) implies that $\sigma_i(G) \geq \sigma_i(G \setminus \{e\})$ for all $i$ $(i = 1, \ldots, n)$, thus we have

$$\begin{aligned}
IE(G)^2 &= \sum_i \sigma_i^2(G) + 2\sum_{i<j} \sigma_i(G)\sigma_j(G) \\
&= trace(I(G)I(G)^t) + 2\sum_{i<j} \sigma_i(G)\sigma_j(G) \\
&= 2 + trace(I(G \setminus \{e\})I(G \setminus \{e\})^t) + 2\sum_{i<j} \sigma_i(G)\sigma_j(G) \\
&= 2 + \sum_i \sigma_i^2(G \setminus \{e\}) + 2\sum_{i<j} \sigma_i(G)\sigma_j(G) \quad \text{(by Eq. (4))} \\
&\geq 2 + \sum_i \sigma_i^2(G \setminus \{e\}) + 2\sum_{i<j} \sigma_i(G \setminus \{e\})\sigma_j(G \setminus \{e\}) \quad \text{(by Eq. (2))} \\
&= 2 + IE(G \setminus \{e\})^2 \ .
\end{aligned}$$

Moreover Eq. (4) implies that, there exists some $i$ such that $\sigma_i(G) > \sigma_i(G \setminus \{e\})$. If $I(G)$ has at least two non-zero singular values and $\sigma_k(G) > \sigma_k(G \setminus \{e\})$ for $1 \leq k \leq n$, then

$$\sum_{i<j} \sigma_i(G)\,\sigma_j(G) = \sigma_1(G)\,\sigma_k(G) + \sum_{i<j,(i,j)\neq(1,k)} \sigma_i(G)\,\sigma_j(G) \ .$$

So $\sigma_1(G)\sigma_k(G) \neq 0$ which leads to

$$\sum_{i<j} \sigma_i(G)\,\sigma_j(G) > \sum_{i<j} \sigma_i(G \setminus \{e\})\,\sigma_j(G \setminus \{e\}) \ .$$

Therefore in Eq. (3) the equality does not occur, if the incidence matrix of the graph $G$ has more than one non-zero singular value. Because $rank(I(G)I(G)^t) = rank(I(G))$. So in the equality case, $rank(I(G))$ must be equal to 1. Finally, if the graph $G$ has more than one edge, $rank(I(G)) > 1$. $\square$

## EXAMPLES

We are going to obtain singular values of $S_n$. The incidence matrix of $S_n$ can be represented in the form of $\begin{bmatrix} I_{n-1} \\ j^t \end{bmatrix}$ where $j$ is the all ones vector, and so

$$I(S_n)I(S_n)^t = \begin{bmatrix} I_{n-1} & j \\ j^t & n-1 \end{bmatrix}.$$

According to the Theorem (6), one can see that the eigenvalues of $I_{n-1}$ interlace the eigenvalues of $I(S_n)I(S_n)^t$, precisely $\lambda_i(I(S_n)I(S_n)^t) \geq 1 \geq \lambda_{i+1}(I(S_n)I(S_n)^t)$ for all $i = 1, \ldots, n-1$. Therefore $\lambda_i(I(S_n)I(S_n)^t) = 1$ for every $i$, $2 \leq i \leq n-1$. Since the rank of $I(S_n)I(S_n)^t$ is equal to $n-1$, then $\lambda_n(I(S_n)I(S_n)^t) = 0$. Also $trace(I(S_n)I(S_n)^t) = 2n-2$ which implies that $\lambda_1(I(S_n)I(S_n)^t) = n$. Then we have $IE(S_n) = \sqrt{n} + n - 2$.

**Question 13** *If $T$ is a tree with $n$ vertices which is not $S_n$, then is it true that $IE(T) > \sqrt{n} + n - 2$?*

**Theorem 14** *Let $T$ be a tree with $n$ vertices, which is not $P_n$, then $IE(T) < IE(P_n)$.*

**Proof.** By applying the statements of [6, P.202] and $\widehat{P_n} = P_{2n-1}$, we have

$$IE(T) = \frac{E(\widehat{T})}{2} < \frac{E(\widehat{P_n})}{2}$$

Because $\widehat{T}$, is a tree of order $2n-1$ which is not $P_{2n-1}$. □

The followings show that there exist some graphs whose incidence energy would be equal to or less than their energy. Although almost for every graph the incidence energy is greater than its energy.

**Proposition 15** *$IE(C_{2k+1}) = E(C_{2k+1})$, for $k \geq 1$.*

**Proof.** The spectrum of $C_n$ is $2\cos(\frac{2\pi i}{n})$, where $i = 1, \ldots, n$. Thus

$$IE(C_n) = 2\sum_{i=1}^{n} \left|\cos\left(\frac{\pi i}{n}\right)\right| .$$

Also for any odd number $n = 2k+1$, we have

$$\begin{aligned} E(C_n) &= 2\sum_{i=1}^{n}\left|\cos\left(\frac{2\pi i}{n}\right)\right| = 2\sum_{i=1}^{k}\left|\cos\left(\frac{2\pi i}{n}\right)\right| + 2\sum_{i=1}^{k+1}\left|\cos\left(\pi + \frac{(2i-1)\pi}{n}\right)\right| \\ &= 2\sum_{i=1}^{n}\left|\cos\left(\frac{\pi i}{n}\right)\right| . \end{aligned}$$

So for odd $n$, $IE(C_n) = E(C_n)$. □

**Question 16** *If $G$ is a connected graph. Then is it true that, if $IE(G) = E(G)$ then $G$ is either an odd cycle or an empty graph?*

**Proposition 17** $IE(C_{4k+2}) < E(C_{4k+2})$, *for* $k \geq 1$.

**Proof.** According to the proof of the previous proposition we have

$$E(C_{4k+2}) = 2\sum_{i=1}^{4k+2}\left|\cos\frac{2\pi i}{4k+2}\right| = 4\sum_{i=1}^{2k+1}\left|\cos\frac{\pi i}{2k+1}\right| = 2IE(C_{2k+1})$$

and we have

$$IE(C_{4k+2}) = 2\sum_{i=1}^{4k+2}\left|\cos(\frac{\pi i}{4k+2})\right| = \frac{2\sin\frac{\pi}{4k+2}}{1-\cos\frac{\pi}{4k+2}}$$

also

$$IE(C_{2k+1}) = 2\sum_{i=1}^{2k+1}\left|\cos\frac{\pi i}{2k+1}\right| = \frac{2}{\sin\frac{\pi}{4k+2}} .$$

Then it follows that $IE(C_{4k+2}) < 2IE(C_{2k+1})$. □

## Acknowledgements

The authors thank the referee for his constructive comments and Professor S. Akbari for continuous encouragement. The second author also indebted to the Institute for Studies in Theoretical Physics and Mathematics (IPM) for supports (Grant No. 87200116).

## References

[1] I. Gutman, The energy of a graph, *Ber. Math.–Statist. Sekt. Forschungsz. Graz* **103** (1978) 1–22.

[2] A. Graovac, I. Gutman, N. Trinajstić, *Topological Approach to the Chemistry of Conjugated Molecules*, Springer–Verlag, Berlin, 1977.

[3] I. Gutman, O. E. Polansky, *Mathematical Concepts in Organic Chemistry*, Springer–Verlag, Berlin, 1986.

[4] I. Gutman, Total $\pi$-electron energy of benzenoid hydrocarbons, *Topics Curr. Chem.* **162** (1992) 29–63.

[5] I. Gutman, Topology and stability of conjugated hydrocarbons. The dependence of total $\pi$-electron energy on molecular topology, *J. Serb. Chem. Soc.* **70** (2005) 441–456.

[6] I. Gutman, The energy of a graph: Old and new results, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), *Algebraic Combinatorics and Applications*, Springer–Verlag, Berlin, 2001, pp. 196–211.

[7] B. Zhou, I. Gutman, J. A. de la Peña, J. Rada, L. Mendoza, On spectral moments and energy of graphs, *MATCH Commun. Math. Comput. Chem.* **57** (2007) 183–191.

[8] L. Ye, X. Yuan, On the minimal energy of trees with a given number of pendent vertices, *MATCH Commun. Math. Comput. Chem.* **57** (2007) 193–201.

[9] H. S. Ramane, H. B. Walikar, Construction of equienergetic graphs, *MATCH Commun. Math. Comput. Chem.* **57** (2007) 203–210.

[10] Y. Hou, Z. Teng, C. Woo, On the spectral radius, $k$-degree and the upper bound of energy in a graph, *MATCH Commun. Math. Comput. Chem.* **57** (2007) 341–350.

[11] H. Hua, On minimal energy of unicyclic graphs with prescribed girth and pendent vertices, *MATCH Commun. Math. Comput. Chem.* **57** (2007) 351–361.

[12] L. Xu, Y. Hou, Equienergetic bipartite graphs, *MATCH Commun. Math. Comput. Chem.* **57** (2007) 363–370.

[13] I. Gutman, S. Zare Firoozabadi, J. A. de la Peña, J. Rada, On the energy of regular graphs, *MATCH Commun. Math. Comput. Chem.* **57** (2007) 435–442.

[14] H. Hua, Bipartite unicyclic graphs with large energy, *MATCH Commun. Math. Comput. Chem.* **58** (2007) 57–83.

[15] I. Gutman, B. Furtula, H. Hua, Bipartite unicyclic graphs with maximal, second–maximal, and third–maximal energy, *MATCH Commun. Math. Comput. Chem.* **58** (2007) 85–92.

[16] Y. Yang, B. Zhou, Minimal energy of bicyclic graphs of a given diameter, *MATCH Commun. Math. Comput. Chem.* **59** (2008) 321–342.

[17] I. Gutman, N. M. M. de Abreu, C. T. M. Vinagre, A. S. Bonifácio, S. Radenković, Relation between energy and Laplacian energy, *MATCH Commun. Math. Comput. Chem.* **59** (2008) 343–354.

[18] Z. Liu, B. Zhou, Minimal energies of bipartite bicyclic graphs, *MATCH Commun. Math. Comput. Chem.* **59** (2008) 381–396.

[19] S. Li, X. Li, On tetracyclic graphs with minimal energy, *MATCH Commun. Math. Comput. Chem.* **60** (2008) 395–414.

[20] V. Nikiforov, The energy of graphs and matrices, *J. Math. Anal. Appl.* **326** (2007) 1472–1475.

[21] V. V. Prasolov, *Problems and Theorems in Linear algebra*, American Mathematical Society, 1994.

[22] R. B. Bapat, S. Pati, Energy of a graph is never an odd integer, *Bull. Kerala Math. Assoc.* **1** (2004) 129–132.

[23] S. Pirzada, I. Gutman, Energy of a graph is never the square root of an odd integer, *Appl. Anal. Discr. Math.* **2** (2008) 118–121.

[24] S. Fajtlowicz, On the conjuctures of Graffiti II, *Congr. Numer.* **60** (1987) 189–197.

[25] H. Sachs, Über Teiler, Faktoren und charakteristische Polynome von Graphen, Teil II, *Wiss. Z. TH Ilmenau* **13** (1967) 405–412.

[26] W. H. Haemers, *Eigenvalue techniques in design and graph theory*, Ph.D. thesis, Eindhoven University of Technology, 1979.

[27] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs – Theory and Application*, third ed., Johann Ambrosius Barth. Heidelberg, 1995.

[28] J. Day, W. So, Singular value nequality and graph energy change, *Electon. J. Lin. Algebra* **16** (2007) 291–299.